Eigenvalues and Eigenvectors

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Characteristic Equation

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Correspondingly, the equation

$$det(A - xI_n) = 0$$

is called the **Characteristic equation** of A.

An example

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Then the characteristic polynomial of A is given by the determinant of the matrix $(A - xI_3)$

$$= det \begin{pmatrix} 1-x & -1 & 0 \\ 1 & 2-x & -1 \\ 3 & 2 & -2-x \end{pmatrix}$$
$$= (1-x)(x^2-2)+(1-x) = (1-x)(x^2-1) = -x^3+x^2+x-1.$$

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Thus the characteristic equation of A is

$$(1-x)(x^2-1) = 0$$

Characteristic Polynomial: Properties

So the characteristic polynomial of an $n \times n$ matrix A, given by $det(A - xI_n)$, is an n-degree polynomial of the form $c_0 + c_1x + c_2x^2 + \ldots + c_nx^n$, where the c_i's belong to the field from which the entries of A are coming.

It is easy to see, that $c_n = (-1)^n$.

Also, c_0 is obtained by putting $x = 0$ in the polynomial, so $c_0 = det(A)$.

And also, $c_{n-1} = (-1)^{n-1} trace(A)$.

Cayley-Hamilton theorem

Theorem

Every square matrix satisfies its own characteristic equation.

So if characteristic equation of a matrix A is $c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n = 0$, then we will have that

$$c_0 I_n + c_1 A + c_2 A^2 + \ldots + c_n A^n = O_n$$

For example, we can check that in the previous example, we will have $-A^3 + A^2 + A - I_3 = O_3$.

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Let $c_0 + c_1x + c_2x^2 + \ldots + c_nx^n = 0$ be the characteristic equation of A, then by Cayley-Hamilton theorem we have

$$c_0I_n + c_1A + c_2A^2 + \ldots + c_nA^n = O_n$$

$\implies -(c_1A + c_2A^2 + \ldots + c_nA^n) = c_0I_n$
$\implies -c_0^{-1}(c_1I_n + c_2A + \ldots + c_nA^{n-1})A = I_n$
(note that $c_0 = det(A) \neq 0$).

Thus $A^{-1} = -c_0^{-1}(c_1I_n + c_2A + \ldots + c_nA^{n-1})$

Example

In the previous example, characteristic equation was $-x^3 + x^2 + x - 1 = 0$, so we have that $-A^3 + A^2 + A - I_3 = O_3$.

Thus $(-A^2 + A + I_3)A = I_3$, which gives that

$$A^{-1} = -A^2 + A + I_3$$

Eigenvalue

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For example, the eigenvalues of $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{pmatrix}$ are $1, -1, 1$, since the characteristic equation is $(1 - x)(x^2 - 1) = 0$.

Eigenvalue

If A is a $n \times n$ matrix, then its characteristic polynomial $p(x)$ is an n-degree polynomial, and hence it has n roots. Thus an $n \times n$ matrix has exactly n eigenvalues. However, the eigenvalues may not be all distinct, as seen in the previous example.

Eigenvalue

If A is a $n \times n$ matrix, then its characteristic polynomial $p(x)$ is an n-degree polynomial, and hence it has n roots. Thus an $n \times n$ matrix has exactly n eigenvalues. However, the eigenvalues may not be all distinct, as seen in the previous example.

A real matrix may have complex eigenvalues, e.g., the matrix $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has the characteristic polynomial $x^2 + 1$. So its eigenvalues are $i, -i$.

Eigenvalues: Properties

Theorem

The product of all the eigenvalues of a matrix is equal to its determinant.

Proof.

Let $c_0 + c_1x + c_2x^2 + \ldots c_nx^n = 0$ be the char. eqn. of a matrix A. Then the product of the roots of this equation is $(-1)^n \frac{c_0}{c_n} = (-1)^n \frac{det(A)}{(-1)^n} = det(A)$. $\square$

Corollary

A matrix is non-invertible if and only if 0 *is an eigenvalue of that matrix.*

Proof.

This happens because a matrix is non-invertible iff $det(A) = 0$. $\square$

Eigenvalues: Properties

Theorem

The eigenvalues of a diagonal matrix is its diagonal elements.

Theorem

If c is an eigenvalue of an invertible matrix A, then $\frac{1}{c}$ is an eigenvalue of A^{-1}.

Proof.

Let A be an $n \times n$ matrix. As c is an eigenvalue, we have $det(A - cI_n) = 0$.
Now $det(A^{-1} - \frac{1}{c}I_n) = \frac{1}{det(A)}det(AA^{-1} - \frac{1}{c}AI_n) = \frac{1}{det(A)}det(I_n - \frac{1}{c}A) = \frac{1}{det(A)}\frac{1}{c^n}det(cI_n - A) = 0.$

Hence $\frac{1}{c}$ is a root of $det(A^{-1} - xI_n) = 0$, and thus it is an eigenvalue of A^{-1}. $\square$

Eigenvectors

Definition

Let A be an $n \times n$ matrix with entries from a field F. A non-null vector $v \in F^n$ is called an eigenvector of A if there exists $c \in F$ such that $Av = cv$. In fact, in this case, c is seen to be an eigenvalue of A and v is called an eigenvector of A corresponding to c.

$Av = cv \implies (A - cI_n)v = \theta$, which shows that the system of equation $(A - cI_n)X = \theta$ has a non-null solution, which happens if and only if $det(A - cI_n) = 0$, i.e., c is an eigenvalue of A.

Eigenvectors: Properties

Theorem

Let $A \in M_{n\times n}(F)$. To an eigenvector of A corresponds a unique eigenvalue of A.

Proof.

If possible let an eigenvector v correspond to two distinct eigenvalues c_1, c_2 of A. Then $Av = c_1 v = c_2 v$, which gives $(c_1 - c_2)v = \theta$. However this is not possible as $c_1 - c_2 \neq 0, v \neq \theta$. $\square$

Theorem

Let $A \in M_{n \times n}(F)$. To each eigenvalue of A, there corresponds at least one eigenvector.

Proof.

If c is an eigenvalue, then $det(A - cI_n) = 0$, so $(A - cI_n)X = \theta$ for some non-null $X \in F^n$. So $Ax = cX$. Thus X is an eigenvector of A corresponding to c. □

Clearly, if v is an eigenvector corresponding to c, any scalar multiple kv is also an eigenvector corresponding to c (as $A(kv) = k(Av) = k(cv) = c(kv)$).

Finding eigenvectors

Let $A = \begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}$. The characteristic equation is $(x+1)(x-7) = 0$, so the eigenvalues are $-1, 7$.

Let $X_1 = (x, y)$ be an eigenvector corresponding to -1, then we have $(A + I_2)X_1 = \theta$, which gives $2x + 3y = 0, 4x + 6y = 0$. So the eigenvectors are given by the set $\{k(-\frac{3}{2}, 1) \mid k \in \mathbb{R}\}$.

Similarly let $X_2 = (x, y)$ be an eigenvector corresponding to 7. Then $(A - 7I_2)X_2 = \theta$, which gives $-6x + 3y = 0, 4x - 2y = 0$. So the eigenvectors corresponding to 7 are given by the set $\{k(1, 2) \mid k \in \mathbb{R}\}$.

Eigenvectors: Properties

Theorem

Two eigenvectors corresponding to two distinct eigenvalues of a matrix are linearly independent.

Proof.

Let v_1, v_2 be two eigenvectors corresponding to two distinct eigenvalues d_1, d_2 of a matrix A. Let $c_1v_1 + c_2v_2 = \theta$. Multiplying by A, we get $c_1Av_1 + c_2Av_2 = \theta$, i.e., $c_1d_1v_1 + c_2d_2v_2 = \theta$. This gives that $d_1(-c_2v_2) + d_2c_2v_2 = \theta$ or $c_2(d_1 - d_2)v_2 = \theta$. As v_2 is non-null and $d_1 \neq d_2$, we must have $c_2 = 0$. Thus $c_1v_1 = \theta$, which gives $c_1 = 0$. This shows that v_1, v_2 are linearly independent. $\square$

Eigenvectors: Properties

Theorem

Let E_c be the set of all eigenvectors corresponding to any particular eigenvalue c of a matrix $A \in M_n(F)$. Then $S_c = E_c \cup \{\theta\}$ forms a subspace F^n, which is called the eigenspace or characteristic subspace corresponding to c.

Proof.

First of all, $\theta \in S_c$. Now let $v_1, v_2 \in E_c$. Then for $f, g \in F$, $A(fv_1+gv_2) = f(Av_1)+g(Av_2) = f(cv_1)+g(cv_2) = c(fv_1+gv_2)$. So $fv_1 + gv_2 \in S_c$. This shows that S_c is a subspace of F^n. □

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The dimension of S_c is called the **geometric multiplicity** of c in A. Note that both -1 and 7 have geometric multiplicity 1 in the previous example.

The multiplicity of c in the characteristic polynomial of A is called its **algebraic multiplicity**. In the first example, algebraic mul. of -1 is 1, whereas that of 1 is 2.